9.1 Relations and their properties

- Relationships between elements of sets are represented using the structure called a **relation**
- A subset of Cartesian product of the sets
- Example: a student and his/her ID

Binary relation

- The most direct way to express a relationship between elements of two sets is to use **ordered pairs** made up of two related elements
- **Binary relation**: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$
- A binary relation from A to B is a set R of ordered pairs where the 1st element comes from A and the 2nd element comes from B

Binary relation

- aRb denotes that $(a,b) \in R$
- When (a,b) belongs to R, a is said to be related to b by R
- Likewise, **n-ary relations** express relationships among n elements
- Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation of these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation, and n is called its **degree**

Example

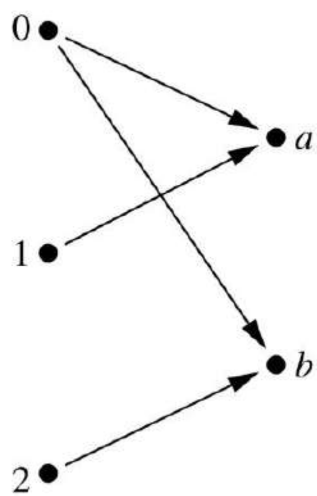
- Let A be the set of students and B be the set of courses
- Let R be the relation that consists of those pairs (a, b) where $a \in A$ and $b \in B$
- If Jason is enrolled only in CSE20, and John is enrolled in CSE20 and CSE21
- The pairs (Jason, CSE20), (John,CSE20), (John, CSE 21) belong to R
- But (Jason, CSE21) does not belong to R

Example

- Let A be the set of all cities, and let B be the set of the 50 states in US. Define a relation R by specifying (a,b) belongs to R if city a is in state b
- For instance, (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in R

Example

- Let A={0, 1, 2} and B={a, b}. Then {(0, a), (0, b), (1, a), (2, b)} is a relation from A to B
- That is 0Ra but not 1Rb



R	a	b
0	×	×
1	×	
2		×

Functions as relations

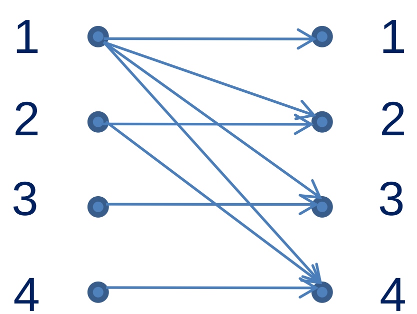
- Recall that a function f from a set A to a set B assigns **exactly one** element of B to each element of A
- The graph of f is the set of ordered pairs (a, b) such that b=f(a)
- Because the graph of f is a subset of A x B, it is a relation from A to B
- Furthermore, the graph of a function has the property that every element of A is the first element of exactly one ordered pair of the graph

Functions as relations

- Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph
- A relation can be used to express one-to-many relationship between the elements of the sets A and B where an element of A may be related to more than one element of B
- A function represents a relation where exactly one element of B is related to each element of A
- Relations are a generalization of functions

Relation on a set

- A relation on the set A is a relation from A to A, i.e., a subset of A x A
- Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R={(a,b)|a divides b}?
- R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}



R	1	2	3	4
1	X	X	X	X
2		X		X
3			X	
4				X

Example

- Consider these relations on set of integers

  $R_1=\{(a,b)|a\leq b\}$

  $R_2=\{(a,b)|a>b\}$

  $R_3=\{(a,b)|a=b \text{ or } a=-b\}$

  $R_4=\{(a,b)|a=b\}$

  $R_5=\{(a,b)|a=b+1\}$

  $R_6=\{(a,b)|a+b\leq 3\}$

  Which of these relations contain each of the pairs (1,1), (1,2), (2,1), (1, -1) and (2, 2)?

- (1,1) is in R_1, R_3, R_4 and R_6; (1,2) is in R_1 and R_6; (2,1) is in R_2, R_5, and R_6; (1,-1) is in R_2, R_3, and R_6; (2,2) is in R_1, R_3, and R_4

Example

- How many relations are there on a set with n elements?
- A relation on a set A is a subset of A x A
- As A x A has n^2 elements, there are 2^{n^2} subsets
- Thus there are 2^{n^2} relations on a set with n elements
- That is, there are $2^{3^2} = 2^9 = 512$ relations on the set {a, b, c}

Properties of relations: Reflexive

- In some relations an element is always related to itself
- Let R be the relation on the set of all people consisting of pairs (x,y) where x and y have the same mother and the same father. Then x R x for every person x
- A relation R on a set A is called **reflexive** if $(a,a) \in R$ for every element $a \in A$
- The relation R on the set A is reflexive if $\forall a((a,a) \in R)$

Example

- Consider these relations on {1, 2, 3, 4}

  R_1={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}

  R_2={(1,1),(1,2),(2,1)}

  R_3={(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}

  R_4={(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}

  R_5={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

  R_6={(3,4)}

  Which of these relations are reflexive?
- R_3 and R_5 are reflexive as both contain all pairs of the (a,a)
- Is the "divides" relation on the set of positive integers reflexive?

Symmetric

- In some relations an element is related to a second element <u>if and only if</u> the 2^{nd} element is also related to the 1^{st} element
- A relation R on a set A is called **symmetric** if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a, b \in A$
- The relation R on the set A is symmetric if $\forall a \, \forall b \, ((a,b) \in R \rightarrow (b,a) \in R)$
- A relation is symmetric if and only if <u>a is related to b</u> implies that <u>b is related to a</u>

Antisymmetric

- A relation R on a set A such that for all a, b ∈ A, if (a, b)∈R and (b, a)∈ R, then a=b is called **antisymmetric**
- Similarly, the relation R is antisymmetric if $\forall a \forall b(((a,b)\in R \land (b,a)\in R) \rightarrow (a=b))$
- A relation is **antisymmetric** if and only if there are no pairs of distinct elements a and b with <u>a related to b</u> and <u>b related to a</u>
- That is, the only way to have <u>a related to b</u> and <u>b related to a</u> is for a and b to be <u>the same</u> element

Symmetric and antisymmetric

- The terms symmetric and antisymmetric are not opposites as a relation can have both of these properties or may lack both of them
- A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a, b) where $a \neq b$

Example

- Consider these relations on {1, 2, 3, 4}

  R_1={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}

  R_2={(1,1),(1,2),(2,1)}

  R_3={(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}

  R_4={(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}

  R_5={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

  R_6={(3,4)}

  R7={(1,1),(2,2),(3,3),(4,4)}

  Which of these relations are symmetric or antisymmetric?

- R_2 and R_3 are symmetric: each (a,b) → (b,a) in the relation
- R_4, R_5, and R_6 are all antisymmetric: no pair of elements a and b with a ≠ b s.t. (a, b) and (b, a) are both in the relation

Example

- Which are symmetric and antisymmetric

  $R_1=\{(a,b)|a\leq b\}$

  $R_2=\{(a,b)|a>b\}$

  $R_3=\{(a,b)|a=b \text{ or } a=-b\}$

  $R_4=\{(a,b)|a=b\}$

  $R_5=\{(a,b)|a=b+1\}$

  $R_6=\{(a,b)|a+b\leq 3\}$
- Symmetric: R_3, R_4, R_6. R_3 is symmetric, if a=b (or a=-b), then b=a (b=-a), R_4 is symmetric as a=b implies b=a, R_6 is symmetric as $a+b\leq 3$ implies $b+a\leq 3$
- Antisymmetric: R_1, R_2, R_4, R_5 . R_1 is antisymmetric as $a\leq b$ and $b\leq a$ imply a=b. R_2 is antisymmetric as it is impossible to have a>b and b>a, R_4 is antisymmtric as two elements are related w.r.t. R_4 if and only if they are equal. R_5 is antisymmetric as it is impossible to have a=b+1 and b=a+1

Transitive

- A relation R on a set A is called **transitive** if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for all $a, b, c \in A$
- Using quantifiers, we see that a relation R is **transitive** if we have

$$\forall a \forall b \forall c\ (((a,b) \in R \land (b,c) \in R) \rightarrow (a,c) \in R)$$

Example

- Which one is transitive?

  $R_1=\{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$

  $R_2=\{(1,1),(1,2),(2,1)\}$

  $R_3=\{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

  $R_4=\{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

  $R_5=\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

  $R_6=\{(3,4)\}$

- R_4 R_5 R_6 are transitive
- R_1 is not transitive as $(3,1)$ is not in R_1
- R_2 is not transitive as $(2,2)$ is not in R_2
- R_3 is not transitive as $(4,2)$ is not in R_3

Example

- Which are symmetric and antisymmetric

  $R_1=\{(a,b)|a\leq b\}$

  $R_2=\{(a,b)|a>b\}$

  $R_3=\{(a,b)|a=b \text{ or } a=-b\}$

  $R_4=\{(a,b)|a=b\}$

  $R_5=\{(a,b)|a=b+1\}$

  $R_6=\{(a,b)|a+b\leq 3\}$
- R_1 is transitive as $a\leq b$ and $b\leq c$ implies $a\leq c$. R_2 is transitive
- R_3 , R_4 are transitive
- R_5 is not transitive (e.g., (2,1), (1,0)). R_6 is not transitive (e.g. (2,1),(1,2))

Combining relations

- Relations from A to B are subsets of $A \times B$, two relations can be combined in any way that two sets can be combined
- Let $A=\{1,2,3\}$ and $B=\{1,2,3,4\}$. The relations $R_1=\{(1,1),(2,2),(3,3)\}$ and $R_2=\{(1,1),(1,2),(1,3),(1,4)\}$ can be combined
- $R_1 \cup R_2=\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(3,3)\}$
- $R_1 \cap R_2=\{(1,1)\}$
- $R_1 - R_2=\{(2,2),(3,3)\}$
- $R_2 - R_1=\{(1,2),(1,3),(1,4)\}$

Example

- Let $R_1=\{(x,y)|x<y\}$ and $R_2=\{(x,y)|x>y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, R_1-R_2, R_2-R_1, and $R_1 \oplus R_2$?
- **Symmetric difference** of A and B: denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B
- We note that $(x,y) \in R_1 \cup R_2$, if and only if $(x,y) \in R_1$ or $(x,y) \in R_2$, it follows that $R_1 \cup R_2=\{(x,y)|x \neq y\}$
- Likewise, $R_1 \cap R_2= \emptyset$, $R_1-R_2=R_1$, $R_2-R_1=R_2$,

  $R_1 \oplus R_2= R_1 \cup R_2-R_1 \cap R_2=\{(x,y)|x \neq y\}$

Composite of relations

- Let R be a relation from a set A to a set B, and S a relation from B to a set C.
- The **composite** of R and S is the relation consisting of ordered pairs (a,c) where $a \in A$, $c \in C$ and for which there exists an element $b \in B$ s.t. $(a,b) \in R$ and $(b,c) \in S$. We denote the **composite of R and S** by $S \circ R$
- Need to find the 2^{nd} element of ordered pair in R the same as the 1^{st} element of ordered pair in S

Example

- What is the composite of the relations R and S, where R is the relation from {1,2,3} to {1,2,3,4} with R={(1,1),(1,4),(2,3),(3,1),(3,4)} and S is the relation from {1,2,3,4} to {0,1,2} with S={(1,0),(2,0),(3,1),(3,2),(4,1)}?
- Need to find the 2^{nd} element in the ordered pair in R is the same as the 1^{st} element of order pair in S
- $S \circ R=\{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$

Power of relation

- Let R be a relation on the set A. The **powers** R^n, $n=1,2,3,\ldots$, are defined recursively by $R^1=R$, $R^{n+1}=R^n \circ R$
- Example: Let $R=\{(1,1),(2,1),(3,2),(4,3)\}$. Find the powers R^n, $n=2,3,4,\ldots$
- $R^2=R\circ R$, we find $R^2=\{(1,1),(2,1),(3,1),(4,2)\}$, $R^3=R^2\circ R$, $R^3=\{(1,1),(2,1),(3,1),(4,1)\}$, $R^4=\{(1,1),(2,1),(3,1),(4,1)\}$.
- It also follows $R^n=R^3$ for $n=5,6,7,\ldots$

Transitive

- Theorem: The relation R on a set A is **transitive** if and only if $R^n \subseteq R$
- Proof: We first prove the "if" part. Suppose $R^n \subseteq R$ for n=1,2,3,… In particular $R^2 \subseteq R$. To see this implies R is transitive, note that if $(a,b) \in R$, and $(b,c) \in R$, then by definition of composition $(a,c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a,c) \in R$. Hence R is transitive

Transitive

- We will use mathematical induction to prove the “only if” part
- Note n=1, the theorem is trivially true
- Assume that $R^n \subseteq R$, where n is a positive integer. This is the induction hypothesis. To complete the inductive step, we must show that this implies that R^{n+1} is also a subset of R
- To show this, assume that $(a,b) \in R^{n+1}$. Because $R^{n+1} = R^n \circ R$, there is an element x with x A s.t. $(a,x) \in R$, and $(x,b) \in R^n$. The inductive hypothesis, i.e., $R^n \subseteq R$, implies that $(x,b) \in R$. As R is transitive, and $(a,x) \in R$, and $(x,b) \in R$, it follows that $(a,b) \in R$. This shows that $R^{n+1} \subseteq R$, completing the proof

9.3 Representing relations

- Can use ordered set, graph to represent sets
- Generally, matrices are better choice
- Suppose that R is a relation from $A=\{a_1, a_2, \ldots, a_m\}$ to $B=\{b_1, b_2, \ldots, b_n\}$. The relation R can be represented by the matrix $M_R=[m_{ij}]$ where

  $m_{ij}=1$ if $(a_i,b_j) \in R$,

  $m_{ij}=0$ if $(a_i,b_j) \notin R$,
- A zero-one (binary) matrix

Example

- Suppose that $A=\{1,2,3\}$ and $B=\{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1=1$, $a_2=2$, and $a_3=3$, and $b_1=1$, and $b_2=2$
- As $R=\{(2,1), (3,1), (3,2)\}$, the matrix R is

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

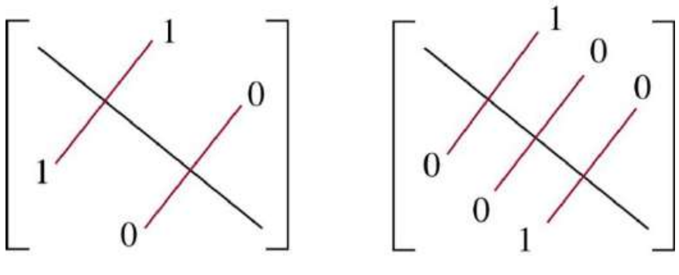
Matrix and relation properties

- The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties
- Recall that a relation R on A is reflexive if (a,a)∈R. Thus R is reflexive if and only if $(a_i,a_i)\in R$ for i=1,2,…,n
- Hence R is reflexive iff $m_{ii}=1$, for i=1,2,…, n.
- R is reflexive if all the elements on the main diagonal of M_R are 1

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$

Symmetric

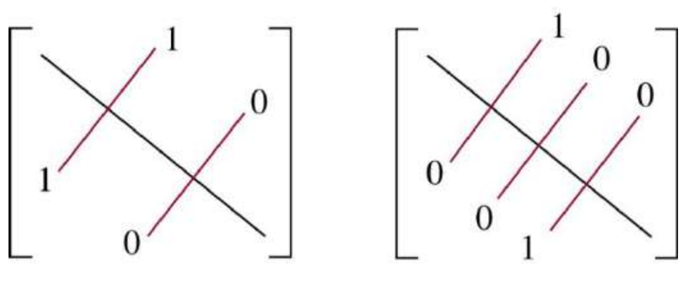
- The relation R is symmetric if $(a,b)\in R$ implies that $(b,a)\in R$
- In terms of matrix, R is symmetric if and only $m_{ji}=1$ whenever $m_{ij}=1$, i.e., $M_R=(M_R)^T$
- R is symmetric iff M_R is a symmetric matrix



(a) Symmetric (b) Antisymmetric

Antisymmetric

- The relation R is symmetric if $(a,b)\in R$ and $(b,a)\in R$ imply $a=b$
- The matrix of an antisymmetric relation has the property that if $m_{ij}=1$ with $i\neq j$, then $m_{ji}=0$
- Either $m_{ij}=0$ or $m_{ji}=0$ when $i\neq j$



(a) Symmetric (b) Antisymmetric

Example

- Suppose that the relation R on a set is represented by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

  Is R reflexive, symmetric or antisymmetric?

- As all the diagonal elements are 1, R is reflexive. As M_R is symmetric, R is symmetric. It is also easy to see R is not antisymmetric

Union, intersection of relations

- Suppose R1 and R2 are relations on a set A represented by M_{R1} and M_{R2}
- The matrices representing the union and intersection of these relations are

$$M_{R1 \cup R2} = M_{R1} \vee M_{R2}$$

$$M_{R1 \cap R2} = M_{R1} \wedge M_{R2}$$

Example

- Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices for $R_1 \cup R_2$ and $R_1 \cap R_2$?

$$M_{R_1 \cup R2} = M_{R_1} \vee M_{R2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad M_{R_1 \cap R2} = M_{R_1} \wedge M_{R2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Composite of relations

- Suppose R is a relation from A to B and S is a relation from B to C. Suppose that A, B, and C have m, n, and p elements with M_S, M_R
- Use Boolean product of matrices
- Let the zero-one matrices for $S \circ R$, R, and S be $M_{S \circ R}=[t_{ij}]$, $M_R=[r_{ij}]$, and $M_S=[s_{ij}]$ (these matrices have sizes $m \times p$, $m \times n$, $n \times p$)
- The ordered pair $(a_i, c_j) \in S \circ R$ iff there is an element b_k s.t.. $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$
- It follows that $t_{ij}=1$ iff $r_{ik}=s_{kj}=1$ for some k

$$M_{S \circ R} = M_R \odot M_S$$

Boolean product (Section 3.8)

- Boolean product A $\odot$ B is defined as

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Replace x with $\wedge$ and + with $\vee$

$$A \cdot B = \begin{bmatrix} (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 1) & (0\wedge 0)\vee(1\wedge 1) \\ (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1\vee 0 & 1\vee 0 & 0\vee 0 \\ 0\vee 0 & 0\vee 1 & 0\vee 1 \\ 1\vee 0 & 1\vee 0 & 0\vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Boolean power (Section 3.8)

- Let A be a square zero-one matrix and let r be positive integer. The r-th Boolean power of A is the Boolean product of r factors of A, denoted by $A^{[r]}$
- $A^{[r]} = \underbrace{A \odot A \odot A \ldots \odot A}_{\text{r times}}$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^{[2]} = A \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{[3]} = A^{[2]} \cdot A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, A^{[4]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Example

- Find the matrix representation of S∘R

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{S \cdot R} = M_R \cdot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Powers R^n

- For powers of a relation

$$M_{R^n} = M_R^{[n]}$$

- The matrix for R2 is

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

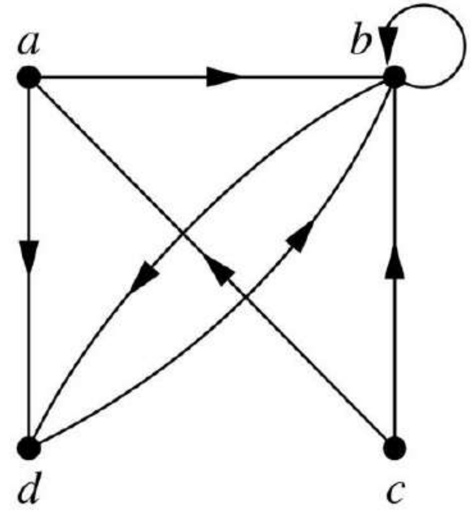
$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing relations using digraphs

- A **directed** graph, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**)
- The vertex a is called the **initial** vertex of the edge (a,b), and vertex b is called the **terminal** vertex of the edge
- An edge of the form (a,a) is called a **loop**

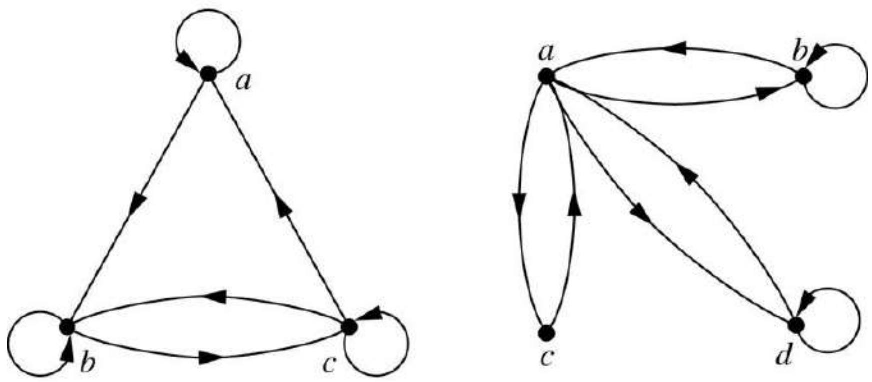
Example

- The directed graph with vertices a, b, c, and d, and edges (a,b), (a,d), (b,b), (b,d), (c,a), (c,b), and (d,b) is shown



$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example



(a) Directed graph of *R*   (b) Directed graph of *S*

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad M_S = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- R is reflexive. R is neither symmetric (e.g., (a,b)) nor antisymmetric (e.g., (b,c), (c,b)). R is not transitive (e.g., (a,b), (b,c))
- S is not reflexive. S is symmetric but not antisymmetric (e.g., (a,c), (c,a)). S is not transitive (e.g., (c,a), (a,b))